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Finite-size corrections for ground states of the *XXZ* Heisenberg chain

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Abstract. The method of de Vega and Woynarovitch is used to calculate finite-size corrections to the ground-state energy in different sectors for the *XXZ* Heisenberg chain. Finite-size scaling amplitudes and correction-to-scaling exponents in the critical region are derived. Using conformal invariance, a scaling dimension $x=(\pi-\gamma)/2\pi$ is extracted corresponding to the electric field operator in the 8-vertex model: this confirms a conjecture of Baxter and Kelland. Finite-size scaling properties near the Kosterlitz–Thouless critical point $\Delta=-1$ are discussed.

1. Introduction

The standard Bethe ansatz method allows one to solve certain models in field theory and statistical mechanics on an infinite lattice (Baxter 1982). In the thermodynamic limit, the boundary conditions can be written as an integral equation with a difference kernel, which may be solved by a Fourier transformation. Unfortunately, however, no such explicit solution has been derived for the finite lattice case.

Recently, a method was given by de Vega and Woynarovitch (1985) for calculating the leading-order finite-size corrections to the ground-state energy of any model which is soluble by the Bethe ansatz. Basically, they show that an integral equation can be written down for the finite lattice case, very similar to the one valid in the thermodynamic limit, but with a correction term involving a sum of delta functions minus its continuum approximation. The asymptotic behaviour of this correction term as the lattice size N goes to infinity can be derived analytically, in favourable cases.

De Vega and Woynarovitch (1985) used a saddle-point method applicable only where the mass gap is non-zero, i.e. the system is non-critical. Here we show, for the example of the *XXZ* Heisenberg–Ising chain, that the leading finite-size corrections can also be derived in the critical region. A brief account of this work was given in Hamer (1985).

After setting up some general equations describing the Heisenberg–Ising model in the remainder of the introduction, we shall turn in § 2 to a discussion of the critical region $-1<\Delta<1$. The finite-size corrections to the ground-state energy for both periodic and antiperiodic boundary conditions will be derived. In § 3 we consider briefly the non-critical region $\Delta\leqslant-1$ and derive scaling functions for the ground states. In § 4 the results are reviewed: critical exponents are extracted from the finite-size scaling amplitudes, results for the finite-size amplitudes and correction-to-scaling

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exponents are tested numerically, and scaling properties in the vicinity of the Kosterlitz–Thouless critical point at $\Delta=-1$ are discussed.

Consider then the XXZ Heisenberg-Ising chain, with Hamiltonian

$$H=-\tfrac{1}{2}\sum_{n=1}^{N}\left(\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y+\Delta\sigma_n^z\sigma_{n+1}^z\right). \tag{1.1}$$

The total number of sites N will be assumed even, for convenience. The Bethe ansatz for this system was discussed in detail by Yang and Yang (1966). The total number m of down spins in the chain is conserved, and so one may label disjoint sectors of states by the quantity

$$y=1-2m/N \tag{1.2}$$

where we assume $m\leqslant N/2$. The Bethe ansatz for the eigenstates involves a momentum p_j for each down spin, and phase factors which depend on pairs of the p_j, given by

$$\theta(p,q)=2\tan^{-1}\left(\frac{\Delta\sin[(p-q)/2]}{\cos[(p+q)/2]-\Delta\cos[(p-q)/2]}\right) \tag{1.3}$$

which is antisymmetric under exchange of p and q. Periodic boundary conditions are satisfied if

$$Np_j=2\pi I_j-\sum_{l=1}^{m}\theta(p_j,p_l) \tag{1.4}$$

where the I_j are integers or half-odd integers, given by

$$I_1,I_2,\ldots,I_m=-\left(\frac{m-1}{2}\right),-\left(\frac{m-1}{2}\right)+1,\ldots,+\left(\frac{m-1}{2}\right) \tag{1.5}$$

for the ground state in each sector. Antiperiodic boundary conditions are satisfied if we then let

$$I_j\to\tilde{I}_j=I_j+\tfrac{1}{2}. \tag{1.6}$$

The energy is given by

$$E=-\frac{N\Delta}{2}+2\sum_{j=1}^{m}(\Delta-\cos p_j) \tag{1.7}$$

while the momentum is

$$P=\sum_{j=1}^{m}p_j=\frac{2\pi}{N}\sum_{j=1}^{m}I_j \tag{1.8}$$

so that the periodic ground state has zero momentum.

2. The region $-1<\Delta<1$

Our main attention will be focused on the periodic ground state in the critical region $-1<\Delta<1$. Then a convenient change of variables is

$$\Delta=-\cos\gamma \qquad 0<\gamma<\pi \tag{2.1}$$

and

$$p=2\tan^{-1}[\cot(\gamma/2)\tanh\lambda] \qquad -\infty<\lambda<\infty. \tag{2.2}$$

Let

$$\phi(\lambda, \gamma) = 2 \tan^{-1}(\cot \gamma \tanh \lambda) \tag{2.3}$$

then (1.4) becomes

$$N\phi(\lambda_j, \gamma/2) = 2\pi I_j + \sum_{i=1}^{m} \phi(\lambda_j - \lambda_i, \gamma) \tag{2.4}$$

and the energy is

$$E = \frac{N}{2} \cos \gamma - \sin \gamma \sum_{j=1}^{m} \phi'(\lambda_j, \gamma/2) \tag{2.5}$$

where the prime denotes differentiation with respect to the λ variable. Note that

$$\phi'\left(\lambda, \frac{\gamma}{2}\right) = \frac{2 \sin \gamma}{\cosh 2\lambda - \cos \gamma}. \tag{2.6}$$

At this point, de Vega and Woynarovitch (1985) proceed to define the function

$$z_N(\lambda, y) = \frac{1}{2\pi}\left(\phi(\lambda, \gamma/2) - \frac{1}{N}\sum_{j=1}^{m} \phi(\lambda - \lambda_j, \gamma)\right). \tag{2.7}$$

This function is continuous and monotonically increasing for real λ; at the roots of (2.4),

$$z_N(\lambda_i, y) = I_i/N. \tag{2.8}$$

The derivative will be denoted

$$\sigma_N(\lambda, y) = \partial z_N/\partial\lambda. \tag{2.9}$$

2.1. *The thermodynamic limit*

When N goes to infinity the λ_i tend to a continuous distribution with density $N\sigma_N(\lambda, y)$:

$$\sigma_\infty(\lambda_i, y) = \lim_{N\to\infty} \frac{1}{N}\left(\frac{I_{i+1} - I_i}{\lambda_{i+1} - \lambda_i}\right) \tag{2.10}$$

and the sum in equation (2.4) reduces to an integral. Following Yang and Yang (1966), we assume that the integration extends between limits $-b_\infty(y)$ to $+b_\infty(y)$ without any gaps, so that taking the difference between successive index values i in equation (2.4) one obtains

$$\sigma_\infty(\lambda, y) = \frac{1}{2\pi}\phi'\left(\lambda, \frac{\gamma}{2}\right) - \int_{-b_\infty(y)}^{b_\infty(y)} \frac{d\mu}{2\pi} \sigma_\infty(\mu, y)\phi'(\lambda - \mu, \gamma) \tag{2.11}$$

where the limit $b_\infty(y)$ is determined by the condition

$$\int_{-b_\infty(y)}^{b_\infty(y)} \sigma_\infty(\lambda, y)\, d\lambda = \lim_{N\to\infty} m/N = \tfrac{1}{2}(1 - y). \tag{2.12}$$

By integrating (2.11) over all λ from $-\infty$ to $+\infty$ (using formulae from appendix 1), we obtain

$$\int_{-\infty}^{\infty} \sigma_\infty(\lambda, y)\, d\lambda = \tfrac{1}{2} + y\left(\frac{1}{2} - \frac{\gamma}{\pi}\right) \tag{2.13}$$

so the 'remainder' terms are

$$\int_{b_\infty(y)}^{\infty} \sigma_\infty(\lambda, y)\,\mathrm{d}\lambda + \int_{-\infty}^{-b_\infty(y)} \sigma_\infty(\lambda, y)\,\mathrm{d}\lambda = \left(1 - \frac{\gamma}{\pi}\right) y. \tag{2.14}$$

The ground-state energy per site is then†

$$f_\infty(y) = \lim_{N\to\infty} \frac{E}{N} = \tfrac{1}{2}\cos\gamma - \sin\gamma \int_{-b_\infty(y)}^{b_\infty(y)} \mathrm{d}\lambda\, \sigma_\infty(\lambda, y)\phi'(\lambda, \tfrac{1}{2}\gamma). \tag{2.15}$$

2.1.1. The case $y = 0$. In this case our system of equations can be solved exactly. From equation (2.14), $b_\infty(y)$ goes to infinity as y goes to zero; then the integral equation (2.11) can be solved by Fourier series expansion to give

$$\sigma_\infty(\lambda, 0) = \frac{1}{2\gamma\cosh(\pi\lambda/\gamma)}. \tag{2.16}$$

The ground-state energy per site is then (Yang and Yang 1966)

$$f_\infty(0) = \tfrac{1}{2}\cos\gamma - \sin^2\gamma \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{\cosh(\pi\lambda)} \frac{1}{[\cosh(2\gamma\lambda) - \cos\gamma]}. \tag{2.17}$$

2.1.2. The case $y = 0+$. When y is small but non-zero, the system of equations has not been solved exactly, but Yang and Yang (1966) were able to analyse them using perturbation methods. They obtained an integral equation for $\sigma_\infty(\lambda, y)$:

$$\sigma_\infty(\lambda, y) = \sigma_\infty(\lambda, 0) + \int_{-\infty}^{-b_\infty(y)} \frac{\mathrm{d}\mu}{\pi} p(\lambda - \mu)\sigma_\infty(\mu, y) + \int_{b_\infty(y)}^{\infty} \frac{\mathrm{d}\mu}{\pi} p(\lambda - \mu)\sigma_\infty(\mu, y) \tag{2.18}$$

where

$$p(\lambda - \mu) = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x\, \exp[\mathrm{i}x(\lambda - \mu)] \frac{\sinh[(\pi - 2\gamma)x/2]}{\sinh[(\pi - 2\gamma)x/2] + \sinh(\pi x/2)}. \tag{2.19}$$

The kernel $p(\lambda - \mu)$ is one that we shall meet frequently hereafter: its relevant properties are listed in appendix 2.

For the energy per site, they found an equation

$$f_\infty(y) - f_\infty(0) = 4\pi\sin\gamma \int_{b_\infty(y)}^{\infty} \sigma_\infty(\lambda, 0)\sigma_\infty(\lambda, y)\,\mathrm{d}\lambda. \tag{2.20}$$

Equation (2.18) can be treated as a perturbed Wiener–Hopf equation, and hence Yang and Yang derive that

$$\exp\left(-\frac{\pi}{\gamma} b_\infty(y)\right) = y\frac{\gamma(\pi - \gamma)}{\pi\tilde{T}_0(0)}[1 + \mathrm{O}(y^2) + \mathrm{O}(y^{4\gamma/(\pi-\gamma)})] \tag{2.21}$$

where $\tilde{T}_0(0)$ is a constant dependent on γ alone. Then

$$f_\infty(y) - f_\infty(0) = \pi\sin\gamma\frac{(\pi - \gamma)}{4\gamma} y^2[1 + \mathrm{O}(y^2) + \mathrm{O}(y^{4\gamma/(\pi-\gamma)})]. \tag{2.22}$$

† This differs by a factor 2 from the quantity $f(\Delta, y)$ defined by Yang and Yang (1966).

2.2. Finite-size corrections

The discussion above is valid in the limit $N \to \infty$. For finite N, de Vega and Woynarovitch (1985) show that one can recast the problem in terms of a similar set of integral equations. From the definitions (2.7) and (2.9), one can write down

$$\sigma_N(\lambda, y) = \frac{1}{2\pi} \phi'\left(\lambda, \frac{\gamma}{2}\right) - \int_{-b_N(y)}^{b_N(y)} \frac{d\mu}{2\pi} \sigma_N(\mu, y) \phi'(\lambda - \mu, \gamma)$$
$$- \int_{-b_N(y)}^{b_N(y)} \frac{d\mu}{2\pi} \phi'(\lambda - \mu, \gamma) \left(\frac{1}{N} \sum_{i=1}^{m} \delta(\mu - \lambda_i) - \sigma_N(\mu, y) \right) \tag{2.23}$$

where the equations corresponding to (2.12)-(2.14) are

$$\int_{-b_N(y)}^{b_N(y)} \sigma_N(\lambda, y) \, d\lambda = m/N = \tfrac{1}{2}(1-y) \tag{2.24}$$

$$\int_{-\infty}^{\infty} \sigma_N(\lambda, y) \, d\lambda = \tfrac{1}{2} + y \left(\frac{1}{2} - \frac{\gamma}{\pi} \right) \tag{2.25}$$

and

$$\int_{b_N(y)}^{\infty} \sigma_N(\lambda, y) \, d\lambda = \left(1 - \frac{\gamma}{\pi} \right) \frac{y}{2}. \tag{2.26}$$

Hence we obtain an integral equation for the difference $\sigma_N(\lambda, y) - \sigma_\infty(\lambda, y)$:

$$\sigma_N(\lambda, y) - \sigma_\infty(\lambda, y) + \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} \phi'(\lambda - \mu, \gamma)(\sigma_N(\mu, y) - \sigma_\infty(\mu, y))$$
$$= - \int_{-b_N(y)}^{b_N(y)} \frac{d\mu}{2\pi} \phi'(\lambda - \mu, \gamma) \left(\frac{1}{N} \sum_{i=1}^{m} \delta(\mu - \lambda_i) - \sigma_N(\mu, y) \right)$$
$$+ 2 \int_{b_N(y)}^{\infty} d\mu \frac{\phi'(\lambda - \mu, \gamma)}{2\pi} \sigma_N(\mu, y)$$
$$- 2 \int_{b_\infty(y)}^{\infty} d\mu \frac{\phi'(\lambda - \mu, \gamma)}{2\pi} \sigma_\infty(\lambda, y). \tag{2.27}$$

This can be manipulated by taking Fourier transforms, or using the operator notation of Yang and Yang (1966), to give

$$\sigma_N(\lambda, y) - \sigma_\infty(\lambda, y) = - \int_{-b_N(y)}^{b_N(y)} \frac{d\mu}{\pi} p(\lambda - \mu) \left(\frac{1}{N} \sum_{i=1}^{m} \delta(\mu - \lambda_i) - \sigma_N(\mu, y) \right)$$
$$+ 2 \int_{b_N(y)}^{\infty} d\mu \frac{p(\lambda - \mu)}{\pi} \sigma_N(\mu, y) - 2 \int_{b_\infty(y)}^{\infty} d\mu \frac{p(\lambda - \mu)}{\pi} \sigma_\infty(\mu, y) \tag{2.28}$$

where $p(\lambda - \mu)$ is given by equation (2.19). For the energy per site we obtain in an analogous way

$$f_N(y) - f_\infty(y) = -2\pi \sin \gamma \left[\int_{-b_N(y)}^{b_N(y)} d\lambda \, \sigma_\infty(\lambda, 0) \left(\frac{1}{N} \sum_{i=1}^{m} \delta(\lambda - \lambda_i) - \sigma_N(\lambda, y) \right) \right.$$
$$\left. + 2 \int_{b_N(y)}^{\infty} d\lambda \, \sigma_\infty(\lambda, 0) \sigma_N(\lambda, y) - 2 \int_{b_\infty(y)}^{\infty} d\lambda \, \sigma_\infty(\lambda, 0) \sigma_\infty(\lambda, y) \right] \tag{2.29}$$

using the relation

$$\int_{-\infty}^{\infty} \frac{d\mu}{\pi} p(\lambda - \mu) \phi'\left(\mu, \frac{\gamma}{2}\right) = \phi'\left(\lambda, \frac{\gamma}{2}\right) - 2\pi\sigma_\infty(\lambda, 0). \tag{2.30}$$

2.2.1. The case $y=0$. In this case equation (2.28) reduces to

$$\sigma_N(\lambda,0)-\sigma_\infty(\lambda,0)=-\int_{-\infty}^{\infty}\mathrm{d}\mu\frac{p(\lambda-\mu)}{\pi}\left(\frac{1}{N}\sum_{i=1}^{m}\delta(\mu-\lambda_i)-\sigma_N(\mu,0)\right)$$
$$=-\frac{1}{\pi}\int_{-P}^{P}\mathrm{d}z\,p(\lambda-\lambda_N(z))\left(\frac{1}{N}\sum_{i=1}^{m}\delta(z-I_i/N)-1\right) \tag{2.31}$$

where $2P=m/N=\frac{1}{2}$ and $\lambda_N(z)$ is the inverse function to $z_N(\lambda,0)$. Now perform a Poisson resummation, by taking the Fourier transform:

$$p(\lambda-\lambda_N(z))=\sum_{n=0}^{\infty}c_n(\lambda)\cos(4\pi nz) \tag{2.32}$$

where

$$c_0(\lambda)=2\int_{-1/4}^{1/4}\mathrm{d}z\,p(\lambda-\lambda_N(z))$$
$$c_n(\lambda)=4\int_{-1/4}^{1/4}\mathrm{d}z\cos(4\pi nz)p(\lambda-\lambda_N(z)) \qquad n>0. \tag{2.33}$$

Then one finds

$$\sigma_N(\lambda,0)-\sigma_\infty(\lambda,0)=-\frac{1}{2\pi}\sum_{j=1}^{\infty}(-1)^{j(m+1)}c_{jm}(\lambda). \tag{2.34}$$

The leading term in this expansion is

$$(-1)^m\frac{2}{\pi}\int_{-1/4}^{1/4}\mathrm{d}z\cos(2\pi Nz)p(\lambda-\lambda_N(z))$$
$$=(-1)^m\frac{2}{\pi}\left(\cos[2\pi Nz_N(\lambda)]\int_{-1/4}^{1/4}\mathrm{d}z\cos[2\pi N(z-z_N(\lambda))]p(\lambda_N(z)-\lambda)\right.$$
$$\left.-\sin[2\pi Nz_N(\lambda)]\int_{-1/4}^{1/4}\mathrm{d}z\sin[2\pi N(z-z_N(\lambda))]p(\lambda_N(z)-\lambda)\right). \tag{2.35}$$

We have not succeeded in evaluating the integral (2.35) in a simple closed form, although approximate expressions can be obtained. Two general features will be needed in the subsequent discussion.

(i) The function $p(\lambda-\lambda_N(z))$ is of bounded variation, as may be seen from the definition (2.19): therefore the Fourier series coefficient (2.35) drops off as $O(1/N)$ at least (Edwards 1979, (2.3.6)).

(ii) As the expression on the right of (2.35) is intended to illustrate, the coefficient is modulated by an oscillatory function in $z_N(\lambda,0)$, with period $(2\pi N)^{-1}$. Similar considerations apply to higher terms in the expansion (2.34). Note that

$$\int_{-\infty}^{\infty}\mathrm{d}\lambda[\sigma_N(\lambda,0)-\sigma_\infty(\lambda,0)]=0. \tag{2.36}$$

Turning to the energy per site, equation (2.29) reduces to

$$f_N(0)-f_\infty(0)=-2\pi\sin\gamma\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\sigma_\infty(\lambda,0)\left(\frac{1}{N}\sum_{i=1}^{m}\delta(\lambda-\lambda_i)-\sigma_N(\lambda,0)\right)$$
$$=-2\pi\sin\gamma\int_{-1/4}^{1/4}\mathrm{d}z\,\sigma_\infty(\lambda_N(z),0)\left(\frac{1}{N}\sum_{i=1}^{m}\delta(z-I_i/N)-1\right). \tag{2.37}$$

Again let us perform a Poisson resummation: write

$$\sigma_\infty(\lambda_N(z), 0) = \sum_{n=0}^{\infty} d_n \cos(4\pi n z). \tag{2.38}$$

Then

$$f_N(0) - f_\infty(0) = -\pi \sin\gamma \sum_{j=1}^{\infty} (-1)^{j(m+1)} d_{jm}. \tag{2.39}$$

Now

$$d_{jm} = 4\int_{-1/4}^{1/4} dz \frac{\cos(2\pi jNz)}{2\gamma\cosh[(\pi/\gamma)\lambda_N(z)]} \qquad j \geqslant 1 \tag{2.40}$$

but

$$\begin{aligned} z_N(\lambda, 0) &= \int_0^\lambda d\mu\, \sigma_\infty(\mu, 0) + \delta_1 z \\ &= \frac{1}{2\pi}\cos^{-1}\left[1/\cosh\left(\frac{\pi\lambda}{\gamma}\right)\right] + \delta_1 z \end{aligned} \tag{2.41}$$

where

$$\delta_1 z = z_N(\lambda, 0) - z_\infty(\lambda, 0). \tag{2.42}$$

Therefore

$$\frac{1}{\cosh[(\pi/\gamma)\lambda_N(z)]} = \cos(2\pi z) + \delta_2 z \tag{2.43}$$

where

$$\delta_2 z = 2\pi \sin(2\pi z)\delta_1 z \tag{2.44}$$

to leading order.

Neglecting the correction term $\delta_2 z$ for the moment, substitution of (2.43) into (2.40) gives

$$d_{jm} \simeq -(-1)^{jm}\frac{2}{\pi\gamma}\left(\frac{1}{j^2N^2-1}\right) \tag{2.45}$$

and therefore

$$f_N(0) - f_\infty(0) \simeq -\pi^2 \sin\gamma/6\gamma N^2 \tag{2.46}$$

which is the desired result. It can also be obtained by direct substitution of (2.43) into (2.37), using (2.16).

It remains to show that the correction term is negligible. Now $\delta_1 z$ is just the integral of the quantity $(\sigma_N(\lambda, 0) - \sigma_\infty(\lambda, 0))$ discussed above: hence we deduce that $\delta_1 z$ is $O(1/N)$, is odd in z, and oscillates with period $(2\pi N)^{-1}$. Its derivative is absolutely continuous; and hence one may infer (e.g. using Edwards (1967), (2.3.5)) that the Fourier integral over $\delta_2 z$ in equation (2.40) is $O(1/N^3)$ and does not affect the leading behaviour given by equation (2.46).

For the antiperiodic ground state, the treatment goes through in very similar fashion. The thermodynamic limit is the same as for the periodic case, so that the ground-state energy per site $\tilde{f}_\infty(0) = f_\infty(0)$. For the finite lattice, we have

$$\tilde{f}_N(0) - \tilde{f}_\infty(0) = -2\pi \sin\gamma \int_{-1/4}^{1/4} \mathrm{d}z\, \sigma_\infty(\lambda_N(z), 0)\left(\frac{1}{N}\sum_{i=1}^{m} \delta(z - \tilde{I}_i/N) - 1\right) \tag{2.47}$$

which is identical to (2.37) except for the replacement $I_i \to \tilde{I}_i$. Substitution of (2.16) and (2.43) into this equation leads to the result

$$\tilde{f}_N(0) - \tilde{f}_\infty(0) \simeq \pi^2 \sin\gamma / 3\gamma N^2. \tag{2.48}$$

Once again, the roots are equally spaced around a circle in the angular variable $\phi = 4\pi z$, and the same arguments as above may be used to show that the corrections to (2.48) are $\mathrm{O}(1/N^3)$.

2.2.2. The case $y = 0+$. We will be particularly interested in the region $y = \mathrm{O}(1/N)$, which will be assumed henceforth. The leading term on the right of equation (2.28) can be analysed in the same way as in the case $y = 0$, giving rise to the same conclusions: $[\sigma_N(\lambda, y) - \sigma_\infty(\lambda, y)]$ is $\mathrm{O}(1/N)$ and oscillates as a function of $z_N(\lambda, y)$, with period $(2\pi N)^{-1}$. The correction terms in (2.28) are also $\mathrm{O}(1/N)$ and are negligible except at large λ. Again we have

$$\int_{-\infty}^{\infty} \mathrm{d}\lambda [\sigma_N(\lambda, y) - \sigma_\infty(\lambda, y)] = 0. \tag{2.49}$$

For the energy per site, the leading term on the right of equation (2.29) can be rewritten

$$f_N(y) - f_\infty(y) \simeq -\frac{\pi \sin\gamma}{\gamma} \int_{-P}^{P} \mathrm{d}z \frac{1}{\cosh[(\pi/\gamma)\lambda_N(z)]}\left(\frac{1}{N}\sum_{i=1}^{m} \delta(z - I_i/N) - 1\right) \tag{2.50}$$

while

$$z_N(\lambda, y) = \frac{1}{2\pi}\cos^{-1}\left(\frac{1}{\cosh(\pi\lambda/\gamma)}\right) + \delta_1 z(y) \tag{2.51}$$

where

$$\delta_1 z(y) = z_N(\lambda, y) - z_\infty(\lambda, 0). \tag{2.52}$$

Therefore

$$\frac{1}{\cosh[(\pi/\gamma)\lambda_N(z)]} = \cos(2\pi z) + \delta_2 z(y) \tag{2.53}$$

where

$$\delta_2 z(y) = 2\pi \sin(2\pi z)\delta_1 z(y) \tag{2.54}$$

to leading order. Neglecting the correction term $\delta_2 z$, and substituting (2.53) into (2.50), one obtains

$$\begin{aligned} f_N(y) - f_\infty(y) &\simeq \frac{\pi \sin\gamma}{\gamma}\left(\frac{1}{\pi} - \frac{1}{N\sin(\pi/N)}\right)\cos\left(\frac{\pi y}{2}\right) \\ &\simeq \frac{-\pi^2 \sin\gamma}{6\gamma N^2} \end{aligned} \tag{2.55}$$

independent of y, to leading order.

It remains to show that the correction terms which we have dropped are negligible.

(i) Write

$$\delta_1 z(y) = [z_N(\lambda, y) - z_\infty(\lambda, y)] + [z_\infty(\lambda, y) - z_\infty(\lambda, 0)]. \tag{2.56}$$

The contribution of the term in $[z_N(\lambda, y) - z_\infty(\lambda, y)]$ to the final result will be $O(N^{-3})$ by the same arguments as were used in the case $y = 0$ and is negligible. The term $[z_\infty(\lambda, y) - z_\infty(\lambda, 0)]$ is $O(y)$, as illustrated by equation (2.13); inserting it into a Fourier integral such as (2.40) gives a result $O(yN^{-2}) = O(N^{-3})$, which is equally small.

(ii) The remaining correction terms on the right of equation (2.29) can be shown quite easily to sum to a result $O(y^2 N^{-1}) = O(N^{-3})$, which is again negligible.

3. Other regions

3.1. *The case $\Delta = -1$*

The case $\Delta = -1$ needs to be analysed separately using rescaled variables (Yang and Yang 1966). But the same chain of argument as given above can be followed, leading to results which are the limiting cases of those given above:

$$f_N(y) - f_\infty(y) = -\pi^2/6N^2 + O(N^{-3}) \tag{3.1}$$

for $y = O(N^{-1})$ and

$$\tilde{f}_N(0) - \tilde{f}_\infty(0) = \pi^2/3N^2 + O(N^{-3}) \tag{3.2}$$

for the antiperiodic ground state.

3.2. *The region $\Delta < -1$*

This case has already been analysed by de Vega and Woynarovitch (1985), but in order to add somewhat to their results, we shall recapitulate the main points. Only the case $y = 0$ will be considered here. The analysis follows very similar lines to those outlined in § 2. The convenient change of variables here is

$$\Delta = -\cosh\gamma \qquad \gamma > 0 \tag{3.3}$$

and

$$p = 2\tan^{-1}[\coth(\gamma/2)\tan\lambda] \qquad -\pi/2 < \lambda < \pi/2. \tag{3.4}$$

Let

$$\phi(\lambda, \gamma) = 2\tan^{-1}(\coth\gamma\tan\lambda). \tag{3.5}$$

Then (2.4) still holds, while the energy is

$$E = \tfrac{1}{2}N\cosh\gamma - \sinh\gamma \sum_{j=1}^{m} \phi'\left(\lambda_j, \frac{\gamma}{2}\right) \tag{3.6}$$

where

$$\phi'\left(\lambda, \frac{\gamma}{2}\right) = \frac{2\sinh\gamma}{\cosh\gamma - \cos 2\lambda}. \tag{3.7}$$

Defining variables $z_N(\lambda)$ and $\sigma_N(\lambda)$ as before, we find

$$\int_{-\pi/2}^{\pi/2} \sigma_\infty(\lambda, 0)\, d\lambda = \tfrac{1}{2} \tag{3.8}$$

and in the thermodynamic limit one obtains an integral equation corresponding to (2.11),

$$\sigma_\infty(\lambda, 0) = \frac{1}{2\pi}\phi'\left(\lambda, \frac{\gamma}{2}\right) - \int_{-\pi/2}^{\pi/2} \frac{d\mu}{2\pi}\sigma_\infty(\mu, 0)\phi'(\lambda-\mu, \gamma) \tag{3.9}$$

with solution

$$\sigma_\infty(\lambda, 0) = \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\frac{\exp(2im\lambda)}{\cosh m\gamma} = \frac{K(k)}{\pi^2}\mathrm{dn}\left(\frac{2K\lambda}{\pi}, k\right) \tag{3.10}$$

where $K(k)$, $\mathrm{dn}(z, k)$ are elliptic functions of modulus k, with $K'(k)/K(k) = \gamma/\pi$. The energy per site in the periodic ground state is then

$$\begin{aligned} f_\infty(0) &= \tfrac{1}{2}\cosh\gamma - \sinh\gamma\int_{-\pi/2}^{\pi/2} d\lambda\, \sigma_\infty(\lambda, 0)\phi'\left(\lambda, \frac{\gamma}{2}\right) \\ &= \tfrac{1}{2}\cosh\gamma - \sinh\gamma\left(1 + 4\sum_{m=1}^{\infty}\frac{1}{\exp(2m\gamma)+1}\right). \end{aligned} \tag{3.11}$$

As regards finite-size corrections, the equation corresponding to (2.37) is

$$f_N(0) - f_\infty(0) = -2\pi\sinh\gamma\int_{-1/4}^{1/4} dz\, \sigma_\infty(\lambda_N(z), 0)\left(\frac{1}{N}\sum_{i=1}^{m}\delta(z - I_i/N) - 1\right). \tag{3.12}$$

Now

$$z_\infty(\lambda, 0) = \int_0^\lambda d\mu\, \sigma_\infty(\mu, 0) = \frac{1}{2\pi}\sin^{-1}\left[\mathrm{sn}\left(\frac{2K\lambda}{\pi}, k\right)\right] \tag{3.13}$$

and therefore

$$\sigma_\infty(\lambda, 0) = (K/\pi^2)(1 - k^2\sin^2(2\pi z))^{1/2}. \tag{3.14}$$

Approximating $\lambda_N(z)$ by $\lambda_\infty(z)$ in (3.12) as usual, we thus obtain

$$\begin{aligned} f_N(0) - f_\infty(0) &\simeq -\frac{2K}{\pi}\sinh\gamma\int_{-1/4}^{1/4} dz (1 - k^2\sin^2(2\pi z))^{1/2} \\ &\quad\times\left(\frac{1}{N}\sum_{i=1}^{m}\delta(z - I_i/N) - 1\right). \end{aligned} \tag{3.15}$$

De Vega and Woynarovitch (1985) performed a Poisson resummation as in § 2, and then applied a saddle-point approximation to the leading term to obtain

$$f_N(0) - f_\infty(0) \simeq -\frac{(8k')^{1/2}}{(\pi N)^{3/2}}K(k)\sinh\gamma\left(\frac{1+k'}{k}\right)^{-N} \tag{3.16}$$

where $k' = (1-k^2)^{1/2}$. The same result may be obtained by contour integration.

We will now obtain a scaling form for this quantity in the vicinity of $\gamma = 0$. The expression (3.15) may be rewritten as

$$f_N(0) - f_\infty(0) \simeq -\frac{2K}{\pi}\sinh\gamma\left(\frac{k}{2N}[S_N(\mu) - S_{N/2}(\mu/2)] - \frac{E(k)}{\pi}\right) \tag{3.17}$$

where $E(k)$ is the elliptic function of the second kind and

$$S_N(\mu) \equiv \sum_{k=0}^{N-1}\left(\frac{\mu^2}{N^2} + 4\sin^2(\pi k/N)\right)^{1/2} \tag{3.18}$$

$$\mu = 2Nk'/k. \tag{3.19}$$

For N large and μ fixed, we have (Hamer and Barber 1981a)

$$S_N(\mu)=\frac{4N}{\pi}+\frac{\mu^2}{2\pi N}\left[\ln\left(\frac{2N}{\pi}\right)+C_{\mathrm{E}}\right]-\frac{\pi}{N}R_{1\frac{1}{2},0}\left(\frac{\mu^2}{4\pi^2}\right)+\frac{\mu}{N}-\frac{\pi}{3N}+\mathrm{O}(N^{-2}) \qquad (3.20)$$

where $R_{1\frac{1}{2},0}(z)$ is a 'remnant function' discussed by Fisher and Barber (1972a) and C_{E} is Euler's constant; while

$$k\simeq 1-\mu^2/8N^2 \qquad (3.21)$$

$$E(k)\simeq 1-\frac{\mu^2}{16N^2}-\frac{\mu^2}{16N^2}\ln\frac{\mu^2}{64N^2} \qquad (3.22)$$

$$(2K(k)/\pi)\sinh\gamma=\pi+\mathrm{O}((\ln N)^{-2}). \qquad (3.23)$$

Hence we find

$$f_N(0)-f_\infty(0)\underset{N\to\infty}{\sim}-(1/N^2)Q_0(\mu)+\mathrm{O}((N\ln N)^{-2}) \qquad (3.24)$$

where

$$Q_0(\mu)=-\frac{\mu^2}{16}+\frac{\mu^2}{8}\left[\ln\left(\frac{\mu}{2\pi}\right)+C_{\mathrm{E}}\right]+\frac{\pi^2}{6}-\frac{\pi^2}{2}R_{1\frac{1}{2},0}\left(\frac{\mu^2}{4\pi^2}\right)+\pi^2R_{1\frac{1}{2},0}\left(\frac{\mu^2}{16\pi^2}\right). \qquad (3.25)$$

Thus we see that the finite-size corrections obey a scaling form in the vicinity of the Kosterlitz-Thouless critical point at $\Delta=-1$ (or $\gamma=0$), up to logarithmic correction terms, with $Q_0(\mu)$ as the scaling function. Taking the limit $\mu\to 0$, which corresponds to $\gamma\to 0$ at fixed N, one finds

$$Q_0(\mu)\underset{\mu\to 0}{\longrightarrow}\tfrac{1}{6}\pi^2 \qquad (3.26)$$

(since $R_{1\frac{1}{2},0}(z)=\mathrm{O}(z^2)$ as $z\to 0$), and hence one regains the result (3.1).

One may perform a similar analysis for the antiperiodic ground state, obtaining the results

$$\tilde{f}_\infty(0)=f_\infty(0) \qquad (3.27)$$

$$\tilde{f}_N(0)-\tilde{f}_\infty(0)\simeq-\frac{2K}{\pi}\sinh\gamma\left(\frac{k}{2N}S_{N/2}(\mu/2)-\frac{E(k)}{\pi}\right) \qquad (3.28)$$

and for N large, μ fixed,

$$\tilde{f}_N(0)-\tilde{f}_\infty(0)\underset{N\to\infty}{\sim}Q_1(\mu)/N^2 \qquad (3.29)$$

$$Q_1(\mu)=\frac{\mu^2}{16}-\frac{\mu^2}{8}\left(\ln\frac{\mu}{8\pi}+C_{\mathrm{E}}\right)+\frac{\pi^2}{3}+\pi^2R_{1\frac{1}{2},0}\left(\frac{\mu^2}{16\pi^2}\right)-\frac{\mu\pi}{2}. \qquad (3.30)$$

4. Discussion

The main result of § 2 was that the finite-size correction to the energy per site of the periodic ground state in the critical region $-1<\Delta<1$ is

$$f_N(y)-f_\infty(y)\underset{N\to\infty}{\sim}-\frac{\pi^2\sin\gamma}{6\gamma N^2} \qquad (4.1)$$

independent of y (to leading order), for $y = O(N^{-1})$. Now the two lowest-lying sectors have $m = N/2$ and $m = N/2 - 1$, or $y = 0$ and $y = 2/N$, respectively. It follows that the mass gap between these two sectors is

$$F_N = N[f_N(2/N) - f_N(0)] \underset{N\to\infty}{\sim} N[f_\infty(2/N) - f_\infty(0)]. \tag{4.2}$$

But the right-hand side of this expression has already been evaluated by Yang and Yang (1966):

$$f_\infty(2/N) - f_\infty(0) = \frac{\pi(\pi-\gamma)\sin\gamma}{\gamma N^2}[1 + O(N^{-2}) + O(N^{-4\gamma/(\pi-\gamma)})] \tag{4.3}$$

and hence

$$F_N \underset{N\to\infty}{\sim} \frac{\pi}{N\gamma}(\pi-\gamma)\sin\gamma. \tag{4.4}$$

The results (4.1) and (4.4) are supported by comparison with some numerical results. Eigenvalues of the Hamiltonian (1.1) have been calculated for even lattices up to $N = 20$ sites and sequence extrapolation methods (Hamer and Barber 1981c) were used to estimate the finite-size scaling amplitudes. The results are compared with the theoretical predictions in table 1, for several values of γ. It can be seen that the agreement is excellent, except for the mass gap amplitude at $\gamma = 0$, where large correction terms render the sequence extrapolation inaccurate.

Table 1. Comparison of theoretical and experimental results for finite-size scaling amplitudes. The amplitude $A_0 = \lim_{N\to\infty}[N^2(f_N(0) - f_\infty(0))]$ and $A = \lim_{N\to\infty}[NF_N]$. The theoretical results are given by equations (4.1) and (4.4) of the text. The 'experimental' numbers were obtained by sequence extrapolation from the numerical values for finite lattices of up to $N = 20$ sites (N even); the approximate error in the final digit is indicated in brackets.

	A_0		A	
γ	Experiment	Theory	Experiment	Theory
$\pi/3$	1.3604 (2)	1.3603	5.4414 (2)	5.4413
$\pi/4$	1.4810 (3)	1.4809	6.6641 (7)	6.6643
$\pi/6$	1.5709 (6)	1.5708	7.852 (1)	7.8540
0	1.6459 (10)	1.6449	9.2 (2)	9.8696

Now Cardy (1984a) has shown by conformal invariance that the finite-size scaling amplitude of the mass gap (or inverse correlation length) for a system at its critical point is related to a critical exponent. If the mass gap scales as

$$F_N \underset{N\to\infty}{\sim} A/N \tag{4.5}$$

where N is the size of the system, then

$$A = 2\pi x \tag{4.6}$$

where x is the scaling dimension of the associated operator. In the Hamiltonian field theory framework, there is a problem in choosing the 'correct' normalisation for the Hamiltonian operator, but von Gehlen *et al* (1986) have pointed out that this may be fixed by looking at the energy-momentum dispersion relation. For the present case, Johnson *et al* (1973) showed that the excitation energy is

$$\Delta E = (\pi/\gamma) \sin\gamma(\sin q_1 + \sin q_2) \tag{4.7}$$

for a two-'particle' excitation with momenta q_1 and q_2, so the Hamiltonian (1.1) should be divided by a factor $(\pi/\gamma)\sin\gamma$ to give the correct dispersion relation for massless particles in the continuum limit. Hence the scaling dimension corresponding to the amplitude (4.4) is

$$x = (\pi - \gamma)/2\pi. \tag{4.8}$$

The operator to which this dimension belongs will be one which produces a transition between the two states involved, e.g. a 'transverse field' $\Sigma_{n=1}^{N} \sigma_n^x$ added to the Hamiltonian (1.1). Such a Hamiltonian would correspond† to the 8-vertex model in an 'electric' field (Baxter 1982). A strong conjecture for its critical exponent was put forward by Baxter and Kelland (1974):

$$\beta_e = \tfrac{1}{4}(\pi/\gamma - 1). \tag{4.9}$$

Using scaling relations between the exponents, our result (4.8) is found to confirm this conjecture.

Another result of § 2 was that the energy per site for the antiperiodic ground state in the critical region scales as

$$\tilde{f}_N(0) - \tilde{f}_\infty(0) \underset{N\to\infty}{\sim} \frac{\pi^2 \sin\gamma}{3\gamma N^2} \tag{4.10}$$

with $\tilde{f}_\infty(0) = f_\infty(0)$, so that the gap between the antiperiodic and periodic ground states (the 'kink mass') scales as

$$G_N = N[\tilde{f}_N(0) - f_N(0)] \underset{N\to\infty}{\sim} \pi^2 \sin\gamma/2\gamma N. \tag{4.11}$$

Then the corresponding finite-size scaling amplitude for the 'correctly' normalised Hamiltonian is

$$B = \pi/2. \tag{4.12}$$

Now Cardy (1984b) has used duality to argue that the kink mass should have an amplitude $B = \pi\eta$ for the Z_2 and Z_3 models in two dimensions. We conjecture that for the present model $B = 2\pi\eta$, so that the result (4.12) corresponds to the known magnetic exponent $\eta = \frac{1}{4}$ for the 8-vertex model (Baxter 1982). A heuristic argument to support this conjecture is that the 8-vertex model, which reduces to the present 6-vertex or *XXZ* model as one special case, reduces in another special case to a *pair* of identical and independent Ising models (Baxter 1982). In that case the mass gap will be equal to the Ising gap, corresponding to the excitation of one Ising lattice; but the kink mass will be twice the Ising kink mass, because the antiperiodic boundary conditions affect both Ising lattices. Then Cardy's result (1984b) for the Ising (Z_2) model implies our conjecture above. A more rigorous demonstration of this result would be desirable, however.

Next, let us consider correction-to-scaling terms. Assuming that the relationship (4.2) holds beyond the leading term (which we have not proved), the result (4.3) of

† I am indebted to Professor Rodney Baxter for this remark.

Yang and Yang (1966) also gives us the leading correction-to-scaling exponent for the mass gap: in the language of Privman and Fisher (1983)

$$y_3=\begin{cases}-\dfrac{4\gamma}{\pi-\gamma} & \text{for } \pi/3>\gamma>0\\ -2 & \text{for } \pi>\gamma\geqslant\pi/3.\end{cases} \tag{4.13}$$

At the point $\Delta=-1$, which is the endpoint of the critical line where a Kosterlitz-Thouless transition in Δ takes place, the relation (4.2) still holds, and from Yang and Yang (1966) we obtain

$$F_N \underset{N\to\infty}{\sim} (\pi^2/N)+\mathrm{O}[(N\ln N)^{-1}] \tag{4.14}$$

which exhibits the logarithmic corrections to scaling expected at this point. For the ground-state energy $f_N(0)$, or the kink mass, our arguments of § 2 indicate an upper bound on the equivalent correction-to-scaling exponent

$$y_3'\leqslant -1. \tag{4.15}$$

These arguments regarding correction-to-scaling terms are not rigorous and it is again useful to check them numerically. Table 2 shows the results. The agreement with (4.13) is good, especially for the two intermediate γ values. At $\gamma=0$ there is some discrepancy, but this is not unexpected due to the logarithmic correction terms. For the ground state, the exponent y_3' does seem to respect the bound (4.15), possibly saturating it at $\gamma=0$. It is noteworthy that the correction-to-scaling terms are much less important for the ground-state energy or the kink mass than for the mass gap.

Table 2. Comparison of theoretical and experimental results for correction-to-scaling exponents. The exponent y_3' is associated with the ground-state energy per site $\tilde{f}_N(0)$; the exponent y_3 with the mass gap. The theoretical result for y_3 is equation (4.13) of the text; the 'experimental' numbers were found as in table 1.

	y_3'	y_3	
γ	Experiment	Experiment	Theory
$\pi/3$	1.95 (1)	2.0 (3)	2.0000
$\pi/4$	1.6 (2)	1.33 (1)	1.3333
$\pi/6$	1.2 (4)	0.80 (1)	0.8000
0	1.0 (4)	0.2 (1)	0.0000

In § 3 our main object was to extract the scaling function for the finite-size corrections near $\Delta=-1$. It was established that for $\Delta\leqslant-1$, the finite-size corrections to the ground-state energy per site may be written

$$f_N(0)-f_\infty(0) \underset{N\to\infty}{\sim} -(1/N^2)Q_0(\mu)+\mathrm{O}[(N\ln N)^{-2}] \tag{4.16}$$

for μ fixed, where

$$\mu=2Nk'/k\simeq 8N\exp(-\pi^2/2\gamma)\simeq 8N\exp\{-\pi^2/2[-2(1+\Delta)]^{1/2}\}. \tag{4.17}$$

A similar scaling form was found for the kink mass. This is in agreement with theoretical expectations (Fisher 1971, Fisher and Barber 1972b, Hamer and Barber 1981a) that the scaling function near the critical point should be a function of the variable $N/\xi = NF$, where in the present case the mass gap for $\Delta \leqslant -1$ is (Johnson *et al* 1973)

$$F \simeq 8\pi \exp(-\pi^2/2\gamma). \tag{4.18}$$

Finally, let us remark on the Callan-Symanzik beta function for this model, which is often used in finite-size scaling analyses to estimate the correlation length exponent ν (Barber 1983). Suppose we introduce a physical lattice spacing a and a 'physical' Hamiltonian H_{phys} related to the lattice Hamiltonian (1) by

$$H_{\text{phys}} = (1/a)H. \tag{4.19}$$

Now suppose we change the spacing a and simultaneously renormalise the coupling Δ so as to keep the physical mass gap F/a constant. Then the beta function may be defined as

$$\beta(\Delta) = a\, \mathrm{d}\Delta/\mathrm{d}a|_{F/a=\text{constant}} \tag{4.20}$$

and one may show quite simply that it is given by

$$\beta(\Delta) = F(\Delta)/F'(\Delta). \tag{4.21}$$

On a finite lattice, the most naive estimate of the beta function is (Hamer and Barber 1981a)

$$\beta_N^{\text{HB}}(\Delta) = F_N(\Delta)/F'_N(\Delta) \tag{4.22}$$

where $F_N(\Delta)$ is the finite-lattice mass gap; but a more rapidly convergent estimate may be obtained from a pair of lattices of different sizes N and N', according to a neat renormalisation group argument of Roomany and Wyld (1980):

$$\beta_{N,N'}^{\text{RW}}(\Delta) = \frac{2\ln[N'F_{N'}(\Delta)/NF_N(\Delta)]}{\ln[N'/N](\partial/\partial\Delta)\ln[NF_N(\Delta)N'F_{N'}(\Delta)]}. \tag{4.23}$$

For $\Delta \leqslant -1$, the result (4.18) leads to

$$\beta(\Delta) \simeq -\frac{2\gamma^3}{\pi^2} \simeq -\frac{4\sqrt{2}}{\pi^2}(-1-\Delta)^{3/2} \tag{4.24}$$

which is the form of algebraic singularity expected at a Kosterlitz-Thouless transition (Barber 1983). Both the estimates (4.22) and (4.23) will converge to (4.24) in this region. But for $-1 < \Delta < 1$, on the critical line, the expression (4.21) is *undefined.* From the result (4.4), we find that the estimate (4.22) gives

$$\beta_N^{\text{HB}}(\Delta) \underset{N\to\infty}{\sim} -\pi\gamma \simeq -\sqrt{2}\pi(1+\Delta)^{1/2} \tag{4.25}$$

while the estimate (4.23) gives zero, to leading order. The Roomany-Wyld estimate (4.23) is therefore clearly superior for the analysis of a Kosterlitz-Thouless transition such as this, because its continuation beyond the critical point at $\Delta = -1$ is smooth, whereas the Hamer-Barber estimate (4.22) suffers an infinite discontinuity in slope. This explains the poor convergence for β_N^{HB} found in Hamer and Barber (1981b).

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Appendix 1

Two useful results listed by Yang and Yang (1966) are reproduced here:

$$\int_{-\infty}^{\infty} \mathrm{d}\alpha \frac{\exp(\mathrm{i}\alpha\gamma)}{\cosh\alpha - \cos\mu} = \frac{2\pi}{\sin\mu} \frac{\sinh[(\pi-\mu)\gamma]}{\sinh\pi\gamma} \tag{A1.1}$$

$$\int_{-\infty}^{\infty} \mathrm{d}\alpha \frac{\exp(\mathrm{i}\alpha\gamma)}{\cosh\alpha} = \frac{\pi}{\cosh(\pi\gamma/2)}. \tag{A1.2}$$

Appendix 2. Properties of the kernel $p(\lambda)$

This function was also discussed by Yang and Yang (1966), appendix C.

(i) For $-1<\Delta<0$ $\quad p(\lambda)>0$

for $0<\Delta<1$ $\quad p(\lambda)<0.$ (A2.1)

(ii) The behaviour at large λ may be found by contour integration

$$p(\lambda) = \sum_n c_{1n} \exp\left(-\frac{2n\pi}{\pi-\gamma}\lambda\right) + c_{2n}\exp\left(-(2n+1)\frac{\pi}{\gamma}\lambda\right) \qquad (\lambda>0) \tag{A2.2}$$

(for π/γ irrational) so $p(\lambda)$ drops off exponentially at large λ.

(iii) $p(\lambda)$ is symmetric in λ.

(iv) $$\int_{-\infty}^{\infty} p(\lambda)\,\mathrm{d}\lambda = \frac{\pi}{2}\left(\frac{\pi-2\gamma}{\pi-\gamma}\right). \tag{A2.3}$$

Note added. After completion of this work, a paper appeared by Blöte *et al* (1986) showing that the finite-size scaling amplitude of the ground-state energy is proportional to c, the central charge of the Virasoro algebra. They deduced the value of this amplitude for the *XXZ* model from previous work by Takahashi (1973, 1974), using the well known correspondence between a finite system with periodic boundary conditions and one at finite temperature. Their result agrees with ours and corresponds to a central charge $c=1$. Equivalent results for the ground-state amplitude in the special case of the *XXX* model have also been obtained by Avdeev and Dörfel (1986), using essentially the same methods as ours.

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